Economics Working Papers (2002–2016)

Economics

2-10-2012

Unfunded pensions and endogenous labor supply

Torben M. Andersen
University of Aarhus, tandersen@econ.au.dk

Joydeep Bhattacharya
Iowa State University, joydeep@iastate.edu

Unfunded pensions and endogenous labor supply

Abstract
A classic result in dynamic public economics states that there is no welfare rationale for pay-as-you-go (PAYG) pensions in a dynamically-efficient neoclassical economy with exogenous labor supply. Parenthetically, a welfare justification for PAYG pensions exists if the economy is dynamically inefficient. Under a sufficient condition that the old be no less risk-averse than the young, these results extend to an economy with endogenous labor supply.

Keywords
pensions, social security, dynamic efficiency, pay-as-you-go, Diamond model, endogenous labor supply

Disciplines
Economics

Title: Unfunded pensions and endogenous labor supply*[1]

Authors: Torben M. Andersen, Joydeep Bhattacharya

Affiliation: Torben M. Andersen, Department of Economics, University of Aarhus, 8000 Aarhus C, Denmark; Phone: +45 8942 1609; Fax: +45 8613 6334; E-mail: tandersen@econ.au.dk

Affiliation: Joydeep Bhattacharya, Department of Economics, Iowa State University, Ames IA 50011-1070, USA. Phone: (515) 294 5886, Fax: (515) 294 0221; E-mail: joydeep@iastate.edu

Proposed Running Head: pensions and endogenous labor supply

Corresponding author: Joydeep Bhattacharya, Department of Economics, Iowa State University, Ames IA 50011-1070, USA. Phone: (515) 294 5886, Fax: (515) 294 0221; E-mail: joydeep@iastate.edu

Abstract: A classic result in dynamic public economics states that there is no welfare rationale for pay-as-you-go (PAYG) pensions in a dynamically-efficient, overlapping-generations economy with exogenous labor supply. Parenthetically, a welfare justification for PAYG pensions exists if the economy is dynamically inefficient. Under the sufficient condition that the old be no less risk averse than the young, both these results extend to an economy with endogenous labor supply.

Keywords: pensions, social security, dynamic efficiency, endogenous labor supply

JEL Classifications: E6, H3

1. Introduction

Governments in about a hundred and fifty countries around the world offer some kind of old-age pension (social security) to their citizens. Most of these pension programs have a substantial unfunded, pay-as-you-go (PAYG) component: the working young are taxed and the proceeds finance a transfer (pension) to the existing retired elderly (defined benefit). Even though many of these programs have been around for nearly a hundred years and routinely absorb 5-15% of G.D.P, the rationale for their very existence continues to be hotly debated – see Blake (2006) for a detailed discussion.[1]

Among academic economists, this debate started with a classic, justly-venerated result, discussed originally by Aaron (1966) and Samuelson (1975), and conducted within the scope of a two-period overlapping-generations model with neoclassical production ("Diamond (1965) model") where the young supply their labor inelastically, the old are retired, and there is no population growth. In that environment, suppose there is a government that finances a fixed payment (benefit/pension) to each old agent by collecting a lump-sum tax from each contemporaneously-alive young agent. Aaron (1966) and Samuelson (1975) showed that the introduction of such a pension system can improve the stationary welfare of all two-period lived agents if and only if the economy is initially dynamically *inefficient* – the net return on capital is less than zero, the "biological interest rate" (Samuelson, 1958).[2] Incidentally, there is no welfare justification for introducing a PAYG pension scheme if the economy is initially dynamically *efficient.* It is worth noting that, in this environment, these two results are flip sides of the same coin; dynamic inefficiency justifies introducing a PAYG pension scheme and dynamic efficiency challenges it. Henceforth, we term this the Aaron-Samuelson result.

The Aaron-Samuelson result, as exposited in textbooks such as Blanchard and Fischer (1989), is a bit more nuanced in that use is made of Samuelson's correspondence principle to require that the initial stationary state, the starting point of the comparison, be dynamically stable. At a stable steady state, a small increase in the benefit level reduces private capital formation. Such crowding out of private capital is justifiable on welfare grounds if and only if the economy was overaccumulating capital (dynamically inefficient) to begin with.

The Aaron-Samuelson result provides a simple, potentially verifiable condition: a PAYG pension system is socially desirable if and only if the gross return on capital is less than the economy's growth rate. Later work by Abel, Mankiw, Summers and Zeckhauser (1989), and more recently Barbie, Hagedorn, and Kaul (2004), suggested that most developed economies, such as the U.S., are most likely dynamically *efficient*; by implication, a PAYG system in such countries is not desirable at least from the standpoint of simple lifecycle models. Subsequent research has argued that dynamic inefficiency is just one, among a long list of reasons, justifying PAYG pension systems.[3]

Our goal here is to revisit the Aaron-Samuelson result in the Diamond model – the environment studied by Samuelson (1975) and Blanchard and Fischer (1989) – and generalize it along two dimensions: a) allow for endogenous/elastic labor supply, and b) permit a distortionary, payroll tax on young labor income.[4,5] These two generalizations unleash a wide range of general-equilibrium effects and makes the analysis daunting as the ensuing discussion lays bare. If labor supply is fixed, a payroll tax is no different than a lump-sum tax; ceteris paribus, the promise of a pension in the future distorts the young agent's saving decision, causing him to save less. This is the classic crowding-out of private capital, alluded to earlier, that is at the heart of the Aaron-Samuelson result. However, when labor supply is elastic/endogenous, a payroll tax, by lowering the take-home wage, also distorts the labor supply decision. If the substitution effect is dominant, ceteris paribus, the young agent supplies less labor (because leisure is relatively cheaper) and receives less income, which further serves to depress savings. In general equilibrium, this reduction in saving causes the aggregate capital stock to fall. However, since aggregate employment falls too, the question remains: what happens to the capital-labor ratio, and hence, the wage and the interest rate? Evidently, the answer to this question is critical in establishing the direction of the welfare effects of PAYG pensions.

It turns out that a sufficient condition for the introduction of a PAYG system to be welfare enhancing in a dynamically-inefficient economy is that the *capital-labor ratio falls*. Interestingly, under that same condition, the introduction of a PAYG system is welfare reducing in a dynamically-efficient economy. In other words, a sufficient condition for the Aaron-Samuelson result to hold in the more general environment is that the capital-labor ratio falls.[6] So, when does this happen? This is where matters get complicated. To sort things out, we first explore the simpler case of lump-sum tax financing of the pension with elastic labor supply. Here, we show that, introduction of a PAYG pension causes the aggregate capital stock to fall, irrespective of dynamic efficiency. However, aggregate employment rises (falls) in a dynamically-efficient (inefficient) economy. It follows that there is no welfare case for introducing a lump-sum tax financed PAYG pension in a dynamically-efficient economy. But now, the flip side, the case for a PAYG pension in a dynamically-inefficient economy, is no longer clear-cut.

We go on to study the more-difficult case of distortionary tax financing of the pension under elastic labor supply. Here, we are able to show the following: under the fairly-mild restriction that the old be no less risk averse than the young, the Aaron-Samuelson result survives in a Diamond model with elastic labor supply.

That is, *under this one restriction*, there is no welfare case for introducing a payroll tax-financed PAYG pension in a dynamically-efficient elastic labor economy, *and* parenthetically, there *is* a welfare rationale for the pension in an otherwise-identical, dynamically-inefficient economy. It is important to note that the aforementioned restriction is merely a sufficient condition, simple and arguably realistic.

The current paper is closest in spirit to pioneering work by Breyer and Straub (1993), discussed in Blake (2006). Their primary focus is on the following issue. Suppose a comparison of the net return to capital with the net biological interest rate reveals that the PAYG system in place is undesirable. Would abolition of such a system (and replacement by a fully funded system) lead to an intergenerational Pareto improvement considering the fact that the young alive would have paid into the system but would not get anything in return? Breyer and Straub (1993) prove that a necessary condition for such improvement is if, in the process, labor supply is distorted.

They go on to ask: are such static distortions to labor supply enough to justify a transition to the fully funded system? It is here that their focus is aligned with ours as their question may be re-interpreted as indirect interest in the Aaron-Samuelson result for economies with endogenous labor supply. To answer this question, they focus only on steady states and consider the steady-state welfare of a representative two-period lived agent. Additionally, they study a PAYG system in which the contribution by the young is in the form of a distortionary payroll tax on young labor income, and the benefit to the pensioners is actuarially and intergenerationally fair: it is tied to one's labor supply (defined contributions). In this setting, they show that, in a dynamically efficient economy, a sufficient condition for steady-state welfare to fall with the payroll tax rate is that private capital does not increase with the payroll tax rate.

It is instructive to compare our work directly to Breyer and Straub (1993). We study a PAYG system with a fixed common benefit level (defined benefits), i.e., the scheme is not actuarially fair. This permits more direct comparison with the classic statements of the Aaron-Samuelson result. We go further than them because we integrate insights regarding stability of steady states from Nourry (2001). Specifically, we prove that if the initial steady state is dynamically efficient and saddle-point stable, a sufficient condition for our version of the PAYG system to be welfare-undesirable is that the capital-labor ratio does not increase with the benefit level. Perhaps most importantly, Breyer and Straub (1993) are largely silent on the important issue: when does their sufficient condition hold? In comparison, we show that if the old are no less risk-averse than the young, our sufficient condition holds under fairly reasonable restrictions. The upshot is that we can definitively claim to have shown that the classic Aaron-Samuelson result, under a mild condition, extends to economies with endogenous labor supply (something that Breyer and Straub (1993) cannot).

The plan for the rest of the paper is as follows. Section 2 outlines the environment of the model, a generalization of Diamond (1965) to endogenous labor supply. Section 3 describes the perfect-foresight competitive equilibria and their comparative static properties; it also derives a condition for saddle-path stability of a steady state. In Section 4, we derive our main results, while Section 5 concludes. Proofs of all major results are to be found in the appendices.

2. The model

Consider a textbook version of the overlapping-generations model with production due to Diamond (1965) augmented to allow for endogenous labor supply. The economy consists of an infinite sequence of two-period lived overlapping generations, an initial old generation, and an infinitely-lived government. Let $t = 1, 2, ...$ index time. At each date t, a new generation comprised of a continuum of measure one of identical members appears. Each agent is endowed with one unit of labor when young and is retired when old. In addition, the initial old agents are endowed with $K_1 > 0$ units of capital.

There is a single final good produced using a standard neoclassical production function $F(K_t, L_t)$ displaying constant returns, where K_t denotes the capital input and L_t denotes the labor input at t. Let $k_t \equiv K_t/L_t$ denote the capital-labor ratio (capital per young agent). Then, output per young agent at time t is $f(k_t)$ where $f(k_t) \equiv F(K_t/L_t, 1)$ is the intensive production function. We assume that $f' > 0 > f''$, and that the usual Inada conditions hold. The final good can either be consumed in the period it is produced, or it can be stored to yield capital the following period. For reasons of analytical tractability, capital is assumed to depreciate 100% between periods.

Let c_{1t} (c_{2t+1}) denote the consumption of the final good at date t (date $t+1$) by a representative young (old) agent born at t. Let L_t denote the labor supply at date t by a young agent. All such agents have preferences representable by the time-separable utility function

$$U(c_{1t}, c_{2t+1}, L_t) \equiv u(c_{1t}) + v(c_{2t+1}) - d(L_t) \tag{1}$$

where u, v and d are twice continuously differentiable, $u'' < 0 < u'$, $v'' < 0 < v'$, and $d' > 0$, $d'' > 0$. We assume the standard limit conditions (see Nourry, 2001, Assumption 1, for example) that would ultimately preclude corner optima.

Young agents supply labor in competitive labor markets, earning a wage of w_t at time t, where

$$w_t \equiv w(k_t) = f(k_t) - k_t f'(k_t) \tag{2}$$

and $w'(k_t) > 0$. In addition, capital is traded in competitive capital markets, and earns a gross real return of R_{t+1} between t and $t+1$, where

$$R_{t+1} \equiv R\left(k_{t+1}\right) = f'(k_{t+1}) \tag{3}$$

with $R'\left(k_{t+1}\right) < 0$.

The government runs a standard pay-as-you-go (PAYG) pension system in each period. It levies a payroll tax (τ) on each young worker so as to finance a lump-sum transfer of $B \geq 0$ to each of the current retired. The government budget constraint is

$$\tau_t w_t L_t = B \tag{4}$$

The benefit level is taken to be the policy variable (defined benefit scheme), and hence, the payroll tax rate is adjusted using (4) to keep the budget balanced.[7] Since population size does not grow, the net rate of return on the PAYG scheme is zero.[8]

Each young agent, born at date $t \geq 1$, chooses how much to consume in each period of life and how much labor to supply when young, by maximizing $U(c_{1t}, c_{2t+1}, L_t)$ subject to

$$c_{1t} = (1 - \tau_t)\, w_t L_t - S_t \tag{5}$$

$$c_{2t+1} = R_{t+1} S_t + B \tag{6}$$

along with $c_{1t} \geq 0$, $c_{2t+1} \geq 0$ and $L_t \in [0, 1]$; here S denotes individual saving. Each initial old agent faces the following budget constraint: $c_{21} \leq R_1 S_0 + B$.

The optimization problem of young agents, for a given τ_t, can be re-stated as

$$\max_{L \in [0,1],\, S \in (0, (1-\tau) wL)} \Gamma\,(L, S, \tau, w, B) \equiv u\,((1 - \tau)\, wL - S) + v\,(RS + B) - d\,(L)\,,$$

where, for the moment, time subscripts have been removed. The first order conditions to the agent's problem, assuming interior solutions, are given by

$$\Gamma_L \equiv u'\,(\cdot)\,(1 - \tau)\, w - d'\,(\cdot) = 0 \tag{7}$$

$$\Gamma_S \equiv -u'\,(\cdot) + R v'\,(\cdot) = 0. \tag{8}$$

For future use, note

$$\Gamma_{LL} = u''(\cdot)\left((1-\tau)w\right)^2 - d''(\cdot) < 0,$$

$$\Gamma_{LS} = -u''(\cdot)(1-\tau)w > 0,$$

$$\Gamma_{SS} = u''(\cdot) + R^2 v''(\cdot) < 0,$$

$$\Gamma_{SL} = -u''(\cdot)(1-\tau)w > 0.$$

The second order conditions to the agent's problem are satisfied if $\Gamma_{LL} < 0$ (which is true) and if $D \equiv \Gamma_{SS}\Gamma_{LL} - \Gamma_{SL}\Gamma_{LS}$ is positive. Notice,

$$\begin{aligned} D &\equiv \left(u''(\cdot) + R^2 v''(\cdot)\right)\left(u''(\cdot)\left((1-\tau)w\right)^2 - d''(\cdot)\right) - \left(u''(\cdot)(1-\tau)w\right)^2 \\ &= R^2 v''(\cdot)\, u''(\cdot)\left((1-\tau)w\right)^2 - d''(\cdot)\left(u''(\cdot) + R^2 v''(\cdot)\right) > 0 \end{aligned} \quad (9)$$

and hence, the second order conditions are satisfied. For future use, define the Arrow-Pratt measure of relative risk aversion for old and young-age felicity to be

$$\Phi_v \equiv -\frac{c_2 v''(\cdot)}{v'(\cdot)}; \quad \Phi_u \equiv -\frac{c_1 u''(\cdot)}{u'(\cdot)}. \quad (10)$$

Henceforth, we assume the following holds for any $\tau \in [0,1]$:

Assumption A1 $u'(\cdot) + u''(\cdot)(1-\tau)wL \geq 0$

Assumption A2 $\Phi_v \leq 1$

Note Assumption A1 is the well-known condition from static models that the substitution effect of a tax increase dominates the income effect. As noted below, this ensures that optimal labor supply is rising in the wage and falling in the tax rate. Noting that

$$u'(\cdot) + u''(\cdot)(1-\tau)wL = u'(\cdot)\left[1 - \Phi_u + \frac{u''(\cdot)}{u'(\cdot)}S\right],$$

it follows that a *necessary* condition for Assumption A1 to hold is

$$\Phi_u \leq 1. \tag{11}$$

This requires the Arrow-Pratt measure of relative risk aversion for *young*-age felicity to be less than unity, an assumption we maintain below. Assumption A2, in turn, requires the Arrow-Pratt measure of relative risk aversion for *old*-age felicity to be less than unity, which ensures that optimal saving is increasing in its return. It deserves mention that Assumption A2 and (11) are merely sufficient conditions that help to render the ensuing analysis tractable. This fact is explored further in examples below.[9]

For future use, we collect well-known information on a young agent's optimal (partial equilibrium) labor supply and savings in the lemma below. A word on notation: if $Y = Y(x, z)$, then $Y_x \equiv \frac{\partial Y}{\partial x}$.

Lemma 1. *a) A young agent's optimal labor supply is described by*

$$L = L(\tau, B, w, R) \tag{12}$$

where

$$L_\tau < 0, L_B < 0, L_w > 0, L_R > 0, \tag{13}$$

and

b) A young agent's optimal savings function is summarized by

$$S = S(\tau, B, w, R) \tag{14}$$

where

$$S_\tau < 0, S_B < 0, S_w > 0, S_R > 0. \tag{15}$$

The proof of Lemma 1 is in Appendix A. Some intuition for the partial-equilibrium results in Lemma 1 is in order. If the labor supply is fixed, a payroll tax is equivalent to a lump-sum tax. In this case, ceteris paribus, a bigger pension raises the future income of the agent. This distorts his savings decision, causing him to save less. When labor supply is endogenous, a payroll tax, in addition, distorts the labor market decision. That is, ceteris paribus, the higher payroll tax required to finance the bigger pension lowers the after-tax wage, which in turn unleashes income and substitution effects for the labor supply decision. If the substitution effect is dominant (Assumption A1), the young agent supplies less labor (because leisure is relatively cheaper) and receives less income, which further serves to depress savings.

The arguments presented above may be used to foreshadow results to come. In general equilibrium, the aforediscussed reduction in saving causes the aggregate capital stock to fall. However, if the aggregate labor supply falls too, the question remains: what happens to the capital-labor ratio? It turns out that the answer to this question (explored in Section 4 below) is critical in establishing the direction of the welfare effects of PAYG social security.

3. Equilibria

Using the government budget constraint, $\tau_t = B/w_t L_t$, (12), and (14), the (general) equilibrium employment level and capital stock (denoted with a "$\hat{}$" on top) can be written

$$\widehat{L}(B, w_t, R_{t+1}) \equiv L(B/w_t L_t, B, w_t, R_{t+1}) \tag{16}$$

$$\widehat{S}(B, w_t, R_{t+1}) \equiv S(B/w_t L_t, B, w_t, R_{t+1}). \tag{17}$$

Since the aggregate saving of the young at any date becomes the start-of-period capital for the next date, and the size of the young at any date is normalized to unity, we have

$$K_{t+1} = S_t, \tag{18}$$

where S_t is defined in eq. (14) above. Since each young agent supplies L_t units of labor, we have

$$k_{t+1} L_{t+1} = S_t. \tag{19}$$

Perfect-foresight competitive equilibria are sequences $\{k_t\}_{t=2}^{\infty}$ that satisfy (19), given the initial $k_1 > 0$, and (16), (17), (4), (2) and (3). Specifically, they are dynamic sequences $\{k_t\}_{t=1}^{\infty}$ that satisfy

$$k_{t+1}\, \widehat{L}\left(B, w\left(k_{t+1}\right), R\left(k_{t+2}\right)\right) = \widehat{S}(B, w\left(k_t\right), R\left(k_{t+1}\right)) \tag{20}$$

3.1. Steady state. Steady state equilibria are time-invariant sequences, k, that satisfy (20), or more specifically, for given B, satisfy

$$k\,\widehat{L}\left(B, w\left(k\right), R\left(k\right)\right) = \widehat{S}(B, w\left(k\right), R\left(k\right)). \tag{21}$$

The next lemma sets out conditions for the existence and uniqueness of a steady-state equilibrium.

Lemma 2. *(Nourry, 2001) Define*

$$\phi\left(k\right) \equiv k - \frac{\widehat{S}(0, w\left(k\right), R\left(k\right))}{\widehat{L}\left(0, w\left(k\right), R\left(k\right)\right)}. \tag{22}$$

Under the previously-made assumptions, if a) $\lim_{k\to 0}\phi\left(k\right) \leq 0$, *and b)* $\phi'\left(k\right) > 0\,\forall k > 0$, *then there exists a unique steady-state equilibrium,* k.

In general though, as discussed in Nourry (2001), Cazzavillan and Pintus (2004), and Nourry and Venditti (2006), conditions for the existence and uniqueness of a steady-state equilibrium in the Diamond (1965) model with endogenous labor supply, are involved and unintuitive; more so, when $B > 0$. For our purposes, it suffices to assume that a unique steady-state solution to (20) exists.

Also for future reference, note that a steady-state equilibrium, k, will be called dynamically efficient if $R\left(k\right) > 1$, dynamically inefficient if $R\left(k\right) < 1$, and the golden rule if $R(k) = 1$.

3.2. Comparative statics – general equilibrium responses. In a steady-state equilibrium, the aggregate capital stock is given by

$$\widehat{K}(B) \equiv \widehat{S}\left(B,\, w\left(\frac{\widehat{K}(B)}{\widehat{L}(B)}\right),\, R\left(\frac{\widehat{K}(B)}{\widehat{L}(B)}\right)\right), \tag{23}$$

and equilibrium aggregate employment by

$$\widehat{L}(B) \equiv \widehat{L}\left(B,\, w\left(\frac{\widehat{K}(B)}{\widehat{L}(B)}\right),\, R\left(\frac{\widehat{K}(B)}{\widehat{L}(B)}\right)\right), \tag{24}$$

and $\widehat{k} \equiv \widehat{K}/\widehat{L}$.

We proceed to determine the properties of equilibrium optimal savings and labor supply. Note these cannot be read off the individual responses – (13)-(15) discussed above – since those had not taken the government budget constraint into account. Using (4), we can re-write (7) and (8) in steady states as

$$u'\left(w\widehat{L} - B - \widehat{S}\right)\left[w - \frac{B}{\widehat{L}}\right] - d'\left(\widehat{L}\right) = 0 \tag{25}$$

and

$$-u'\left(w\widehat{L}-B-\widehat{S}\right)+R\cdot v'\left(R\widehat{S}+B\right)=0 \tag{26}$$

Hence, totally differentiating (25)-(26) yields

$$\begin{aligned} u''(\cdot)\left[\widehat{L}\partial w+w\partial\widehat{L}-\partial B-\partial\widehat{S}\right]\left[w-\frac{B}{\widehat{L}}\right]+u'(\cdot)\left[\partial w-\frac{1}{\widehat{L}}\partial B+\frac{B}{\widehat{L}^2}\partial\widehat{L}\right]-d''(\cdot)\partial\widehat{L} &= 0 \\ -u''(\cdot)\left[L\partial w+w\partial\widehat{L}-\partial B-\partial\widehat{S}\right]+v'(\cdot)\partial R+Rv''(\cdot)\left[\widehat{S}\partial R+R\partial\widehat{S}+\partial B\right] &= 0. \end{aligned}$$

Eventually, we will require knowledge of these equilibrium responses locally near $B=0$. To that end, evaluating these expressions at $B=0$ yields

$$\left[u''(\cdot)w^2-d''(\cdot)\right]\partial\widehat{L}+\left[-u''(\cdot)w\right]\partial\widehat{S} = \left[-u''(\cdot)w\widehat{L}-u'(\cdot)\right]\partial w+\left[u''(\cdot)w+\frac{u'(\cdot)}{\widehat{L}}\right]\partial B \tag{27}$$

$$\left[-u''(\cdot)w\right]\partial\widehat{L}+\left[u''(\cdot)+R^2v''(\cdot)\right]\partial\widehat{S} = u''(\cdot)\widehat{L}\partial w-\left[u''(\cdot)+Rv''(\cdot)\right]\partial B-\left[v'(\cdot)+Rv''(\cdot)\widehat{S}\right]\partial R \tag{28}$$

From here, the different equilibrium responses are easily computed using Cramer's rule. For future reference, define

$$\Phi_v^0\equiv-\frac{c_2v''(\cdot)}{v'(\cdot)}|_{B=0}\,;\quad \Phi_u^0\equiv-\frac{c_1u''(\cdot)}{u'(\cdot)}|_{B=0}\,, \tag{29}$$

the expressions in (10) evaluated at $B = 0$. Recall under Assumption A2 and (11),

$$\Phi_v^0 \leq 1 \text{ and } \Phi_u^0 \leq 1 \tag{30}$$

are assumed to hold.

We collect information about several general-equilibrium responses in the following lemma. Lemma 3 builds on the results of Lemma 1 by requiring that optimal decisions (in equilibrium) by the agent also respect the government budget constraint. It treats w, R, and B as exogenous variables, and studies the impact (evaluated near $B = 0$) on optimal choices by changing one parameter at a time.

Lemma 3. *Evaluated in equilibrium at $B = 0$,*

a)

$$\widehat{S}_w\Big|_{B=0} > 0; \; \widehat{L}_w\Big|_{B=0} > 0$$

b)

$$\Phi_v^0 < 1 \Longleftrightarrow \mathit{sign} \;\; \widehat{L}_R\Big|_{B=0} = \mathit{sign} \;\; \widehat{S}_R\Big|_{B=0} > 0 \tag{31}$$

c)

$$\widehat{S}_B\Big|_{B=0} < 0,$$

and

d) if $R < 1$,

$$\widehat{L}_B\Big|_{B=0} < 0.$$

A few words about Lemma 3 are in order. Assumption A1 (substitution effect dominates the income effect of a payroll tax increase) ensures that optimal employment increases with the wage. Assumption A2 (substitution effect dominates the income effect of a increase in the interest rate) ensures that optimal saving increases with its return. Ceteris paribus, an increase in B increases future income and depresses saving; it reduces young labor supply if $R < 1$. Why? Ceteris paribus, an increase in future income reduces the need to save (and hence, reduces the desire to earn more current income). When $R < 1$, an unit saved would bring less than one in the future, further depressing the desire to save. When $R > 1$, there would be a countervailing and confounding effect increasing the need to work more when young and save.

3.3. Stability. As discussed by Blanchard and Fischer (1989), Samuelson's "correspondence principle" suggested a tight link between the stability properties of a steady state and its comparative static properties. Since the ultimate goal of the paper is to establish a particular comparative static property, we start off by studying the stability properties of a steady state. To that end, we linearize (20) around a steady state (evaluated near $B = 0$) to get

$$\widehat{L}\widetilde{k}_{t+1} + \widehat{k}\left[\widehat{L}_w w_k \widetilde{k}_{t+1} + \widehat{L}_R R_k \widetilde{k}_{t+2}\right] = \widehat{S}_w w_k \widetilde{k}_t + \widehat{S}_R R_k \widetilde{k}_{t+1},$$

where the tilde over k denotes deviation from its steady-state value, and where X_j denotes the derivative of the function X with respect to the variable j (and $j \neq t+n, n = 0,1,2$). Using $w_k = -kR_k > 0$, and re-organizing the previous equation, we get

$$\widehat{k}\ \widehat{L}_R R_k \widetilde{k}_{t+2} + \left[\left(\widehat{L} - k\ \widehat{L}_w k R_k\right) - \widehat{S}_R R_k\right]\widetilde{k}_{t+1} + \widehat{S}_w \widehat{k} R_k \widetilde{k}_t = 0$$

and finally, $\widetilde{k}_{t+2} + A_1\,\widetilde{k}_{t+1} + A_0\,\widetilde{k}_t = 0$, where

$$A_1 \equiv \frac{\left(\widehat{L} - \widehat{k}\widehat{L}_w \widehat{k} R_k\right) - \widehat{S}_R R_k}{\widehat{k}\ \widehat{L}_R R_k} < 0; \quad A_0 \equiv \frac{\widehat{S}_w}{\widehat{L}_R} > 0.$$

The sign conditions follow from $R_k < 0$, and Lemma 3.

Lemma 4. *a) Evaluated near $B = 0$, a steady state, $\widehat{k}$, is saddle-point stable if*

$$\left[\widehat{L} + \left[\widehat{L}_R\Big|_{B=0} - \widehat{k}\ \widehat{L}_w\Big|_{B=0}\right]\widehat{k}R_k - \widehat{S}_R\Big|_{B=0} R_k + \widehat{S}_w\Big|_{B=0}\widehat{k}R_k\right] > 0 \tag{32}$$

holds.

b) If $\phi'(k) > 0$ holds (where $\phi(k)$ is defined in (22) above), then (32) is satisfied, implying if the steady state k is unique (see Lemma 2), it is saddle-point stable.

This extends the result in Nourry (2001) to the case with a tax-financed pension scheme (defined benefits).

4. Welfare effects of a public pension

The government is assumed to be utilitarian. It determines the benefit level B by maximizing the lifetime utility of an agent in a steady state. The government's objective function, by use of (4), can be written as

$$U(B) = u\left(w\left(\frac{\widehat{K}(B)}{\widehat{L}(B)}\right)\widehat{L}(B) - B - \widehat{K}(B)\right)$$
$$+v\left(R\left(\frac{\widehat{K}(B)}{\widehat{L}(B)}\right)\widehat{K}(B) + B\right) - d\left(\widehat{L}(B)\right),$$

where the general equilibrium employment and capital stock are given by (24) and (23) respectively. A marginal change in the benefit level brings about a change in agent welfare by an amount:

$$U'(B) = u'(\cdot)\left[w\frac{\partial\widehat{L}}{\partial B} + w'\left(\frac{\widehat{K}}{\widehat{L}}\right)\frac{\widehat{L}\partial\left(\frac{\widehat{K}}{\widehat{L}}\right)}{\partial B} - 1 - \frac{\partial\widehat{K}}{\partial B}\right]$$
$$+v'(\cdot)\left[R'\left(\frac{\widehat{K}}{\widehat{L}}\right)\frac{\widehat{K}\partial\left(\frac{\widehat{K}}{\widehat{L}}\right)}{\partial B} + R\frac{\partial\widehat{K}}{\partial B} + 1\right] - d'\left(\widehat{L}\right)\frac{\partial\widehat{L}}{\partial B}.$$

Using the first order conditions (7) and (8) as well as the government budget constraint (4), the above optimality condition can be written compactly as

$$U'(B) = v'(\cdot)\left[(1-R) + (R-1)\ w'\left(\widehat{k}\right)\widehat{L}\ \frac{\partial\widehat{k}}{\partial B} + R\tau w\left(\widehat{k}\right)\ \frac{\partial\widehat{L}}{\partial B}\right]. \tag{33}$$

4.1. General equilibrium effects. Broadly speaking, introducing a unfunded pension or raising the promised benefit level infinitesimally has three effects: (i) savings-effect, (ii) capital-labor substitution effect, and (iii) labor supply effect. The savings effect is the most standard and well-known (since Aaron, 1966). If the initial equilibrium is characterized by $R > 1$ (dynamic efficiency), it follows that an increase in benefits lowers utility since it shifts savings out of capital (with gross return R)

into the PAYG-pension with a gross return of 1. The capital-labor substitution effect emerges because the composition of income according to its source, wages or capital, matters – it matters because wage income is earned when young and capital income when old. If $R = 1$, this composition does not matter. However, if $R > 1$, there is, on the margin, a gain from shifting to wage from capital income since the former can be invested with a return factor $R > 1$. In that case, if a benefit increase leads to an increase in the capital-labor ratio ($\frac{\partial \widehat{k}}{\partial B} > 0$), then such a benefit increase is welfare improving. The opposite holds if $\frac{\partial \widehat{k}}{\partial B} < 0$. Finally, as previously discussed, the labor supply effect arises because the labor supply decision is distorted by the payroll tax – indeed employment is inefficiently low. In this case, if a benefit increase raises employment, the effect on welfare is positive.

It is readily apparent from (33) that the issue of whether public pensions are desirable is not settled simply by dynamic efficiency or inefficiency. In some specific cases, a bit more information can be gleaned from (33). For example, in the case of exogenous labor supply, it is well-known that $\frac{\partial \widehat{k}}{\partial B} < 0$ holds at a stable steady state (see Blanchard and Fischer, 1989). Hence, it follows from (33) that sign $U'(B) =$ sign $(1 - R)$, a restatement of the classic Aaron-Samuelson result. Similarly, additional information on this issue may be obtained near the golden rule. When $R = 1$, i.e., the economy is initially at the golden rule, and labor supply is elastic, increasing the size of public pensions is not welfare neutral because of the income tax distortion to labor supply. If a lump-sum tax is available, then at the golden rule, the optimal size of the public pension is zero. A version of this last statement is also discussed in Blake (2006), chapter 4.

One final point of note. It is clear from (33) that, in general, if $U'(B) < 0$ holds when $R > 1$, it does not follow that $U'(B) > 0$ holds when $R < 1$. That is, if dynamic efficiency implies PAYG pensions are undesirable, then dynamic inefficiency may not imply PAYG pensions are desirable. And yet, as discussed above, in the case of exogenous labor supply, this bi-directional implication of the Aaron-Samuelson result is true.

4.2. Local results. The strategy in the following is to consider the marginal value of introducing a small public pension if there is none to begin with. It follows from (33), for $B = \tau = 0$,

$$U'(0) = v'(\cdot)(1-R)\left[1 - w'\left(\widehat{k}\right)\widehat{L}\left.\frac{\partial \widehat{k}}{\partial B}\right|_{B=0}\right]. \tag{34}$$

It is immediate from (34) that if $R > 1$, a small increase in the size of the pension starting from a size of zero is welfare reducing if it is the case that

$$\left.\frac{\partial \widehat{k}}{\partial B}\right|_{B=0} < 0 \tag{35}$$

holds. In other words, in a dynamically-efficient economy, a necessary and sufficient condition for $U'(0) < 0$ is that $1 - w'\left(\widehat{k}\right)\widehat{L}\frac{\partial \widehat{k}}{\partial B} > 0$; a sufficient condition is that $\left.\frac{\partial \widehat{k}}{\partial B}\right|_{B=0} < 0$. Put differently, there is *no* welfare rationale for introducing a PAYG system in a dynamically-efficient economy if $\left.\frac{\partial \widehat{k}}{\partial B}\right|_{B=0} < 0$, i.e., the pension system *crowds out the equilibrium capital-labor ratio*. Similarly, it follows from (34) that if $R < 1$, a necessary and sufficient condition for $U'(0) > 0$ is that $1 - w'\left(\widehat{k}\right)\widehat{L}\left.\frac{\partial \widehat{k}}{\partial B}\right|_{B=0} > 0$; a sufficient condition is that $\left.\frac{\partial \widehat{k}}{\partial B}\right|_{B=0} < 0$, the same as in (35).[10] In other words, the bi-directional implication of the Aaron-Samuelson result discussed earlier is restored locally near $B = 0$. We summarize the above discussion in Proposition 1 below.

Proposition 1. *A small increase in the size of the pension, starting from a size of zero, is welfare enhancing (reducing) in a dynamically inefficient (efficient) economy if (35) holds in equilibrium.*

A limited version of this result appears in Breyer and Straub (1993). For us, the issue boils down to, when does the sufficient condition (35) hold, the subject matter of the following lemma.

Lemma 5. *a)*

$$\left.\frac{\partial \widehat{k}}{\partial B}\right|_{B=0} = \frac{\left.\widehat{S}_B\right|_{B=0} - \widehat{k}\ \left.\widehat{L}_B\right|_{B=0}}{\widehat{L} - \left[\left.\widehat{S}_w\right|_{B=0} w_k + \left.\widehat{S}_R\right|_{B=0} R_k\right] + \widehat{k}\left[\left.\widehat{L}_w\right|_{B=0} w_k + \left.\widehat{L}_R\right|_{B=0} R_k\right]}.$$

b) If k is saddle-point stable, i.e., (32) holds,

$$sign\ \left.\frac{\partial \widehat{k}}{\partial B}\right|_{B=0} = sign\ \left[\left.\widehat{S}_B\right|_{B=0} - \widehat{k}\ \left.\widehat{L}_B\right|_{B=0}\right],$$

where $\widehat{S}_B$ and $\widehat{L}_B$ are computed in Lemma 3.

The upshot is that use of the stability condition allows us to state a sufficient condition for (35) in terms of the equilibrium saving and labor supply responses to the public pension, i.e.,

$$\left.\frac{\partial \widehat{k}}{\partial B}\right|_{B=0} < 0 \Leftrightarrow \left.\widehat{S}_B\right|_{B=0} - \widehat{k}\ \left.\widehat{L}_B\right|_{B=0} < 0 \tag{36}$$

It is clear from (36) that the magnitude and direction of the saving and labor supply responses (to a change in the pension) will be crucial in what follows.

The thrust of models with exogenous or perfectly inelastic labor supply $\left(\widehat{L}_B = 0\right)$ is that for dynamically-inefficient economies, there is a case for public pensions if private savings (capital) is crowded out ($\widehat{S}_B < 0$). This is the well-known Aaron-Samuelson result. With endogenous or elastic labor supply, however, the crowding out of capital is simply not enough; after all, it may be theoretically possible for labor supply to decrease $\left(\widehat{L}_B < 0\right)$ enough so that the capital-labor ratio increases $\left(\frac{\partial \widehat{k}}{\partial B} > 0\right)$ even though the capital stock falls ($\widehat{S}_B < 0$).

Lump-sum tax. We start our investigation of the elastic labor supply case by studying the simpler case of a lump-sum tax, T. In this case, the first order conditions to the individual's optimization problem (incorporating the government's budget constraint, $T = B$)[11] can be written as

$$w\,u'\left(w\,\widehat{L} - B - \widehat{S}\right) - d'\left(\widehat{L}\right) = 0, \tag{37}$$

$$-u'\left(w\widehat{L} - B - \widehat{S}\right) + R\,v'\left(R\widehat{S} + B\right) = 0. \tag{38}$$

As before, using (37)-(38), we can derive the equilibrium responses of saving and employment to changes in B, and evaluate them near $B = 0$. Let $\left.\frac{\partial \widehat{J}}{\partial B}\right|_{\mathrm{ls}}$ where $\widehat{J} = \widehat{L}, \widehat{S}$ denote a change in the optimal J brought on by a increase in B (evaluated near $B = 0$) when B is financed by a lump-sum tax.

Proposition 2. *For a dynamically-efficient economy, with endogenous labor supply,* *there is no welfare case for introducing a PAYG pension financed by a lump-sum tax.*

The proof is in the appendix and proceeds by proving that under a lump-sum tax financing scheme,

$$\widehat{L}_B\Big|_{\text{ls}} > 0 \text{ if } R > 1, \quad \widehat{S}_B\Big|_{\text{ls}} < 0 \text{ if } R > 1,$$

(reducing capital and raising labor supply), and hence, from (36) it follows that the capital-labor ratio necessarily falls. Accordingly, from (35), there would be no welfare case for a unfunded public pension in this case, as stated in Proposition 2. Note however,

$$\widehat{L}_B\Big|_{\text{ls}} < 0 \text{ if } R < 1, \quad \widehat{S}_B\Big|_{\text{ls}} < 0 \text{ if } R < 1;$$

hence, the sign of $\frac{\partial \widehat{k}}{\partial B}$ is not immediate when $R < 1$. In other words, *the case for a public pension in a dynamically-inefficient economy is not clear-cut even if the pension is financed with a lump-sum tax.*

Payroll tax. More generally, though, as is well known, an income tax releases both a substitution and income effect, and if the former dominates (see Assumption A1), it follows that labor supply decreases with a tax increase, cf Lemma 1, and this goes in the direction of increasing the capital-labor ratio. The distortion arising via the substitution effect thus makes the welfare effect of the public pension more complicated.

As the next lemma shows, relative to when a lump-sum tax is used, $\widehat{L}_B$ is smaller (because of the substitution effect) when a payroll tax replaces the lump-sum tax.

Lemma 6.

$$
\begin{aligned}
\widehat{L}_B\Big|_{B=0} &< \begin{cases} \widehat{L}_B\Big|_{ls} > 0 \ \textit{if}\ R > 1 \\ \widehat{L}_B\Big|_{ls} < 0 \ \textit{if}\ R < 1 \end{cases} \\
\widehat{S}_B\Big|_{B=0} &< \widehat{S}_B\Big|_{ls} < 0 \ \textit{for}\ R \gtrless 1
\end{aligned}
$$

However, at the same time, $\widehat{S}_B$ falls, making it harder to evaluate the overall effect on (36). Using Lemma 6 and Lemma 3, we can re-write (36) as

$$
\begin{aligned}
& \widehat{S}_B\Big|_{B=0} - \widehat{k}\ \widehat{L}_B\Big|_{B=0} \\
= \ & \widehat{S}_B\Big|_{\text{ls}} - \widehat{k}\,\widehat{L}_B\Big|_{\text{ls}} + \frac{u'(\cdot)\frac{1}{\widehat{L}}\left[u''(\cdot)w\right]}{D} - \widehat{k}\frac{u'(\cdot)\frac{1}{\widehat{L}}\left[u''(\cdot)+v''(\cdot)R^2\right]}{D}
\end{aligned} \tag{39}
$$

where $D > 0$. For $R > 1$, Lemma 6 proves that

$$
\widehat{S}_B\Big|_{\text{ls}} - \widehat{k}\,\widehat{L}_B\Big|_{\text{ls}} < 0.
$$

Hence, for a dynamically-efficient economy, it follows from (39) that (36) holds if

$$u'(\cdot)\frac{1}{\widehat{L}}\left[u''(\cdot)w\right] - \widehat{k}\left\{u'(\cdot)\frac{1}{\widehat{L}}\left[u''(\cdot)+v''(\cdot)R^2\right]\right\} < 0. \tag{40}$$

Lemma 7. *a) If*

$$1 \geq \Phi_v^0 \geq \Phi_u^0 \tag{41}$$

(as defined in (29)) holds, then (40) is satisfied.[12,13]

Notice that (30), consequent to Assumptions A1-A2, had already imposed $1 \geq \Phi_v^0$, and $1 \geq \Phi_u^0$. It also deserves mention that both Φ_v and Φ_u are, in general, endogenous variables. The restriction in (41) translates into a parametric condition for the commonly-used CRRA form of time-separable utility.

Lemma 8. *Suppose* $U(c_1, c_2, L) = \frac{1}{1-\gamma}c_1^{1-\gamma} + \beta\frac{1}{1-\rho}c_2^{1-\rho} - \frac{1}{1+\sigma}L^{1+\sigma}$, $\sigma > 0$, $\rho > 0, \gamma > 0, \beta > 0$. *Then (41) holds if* $1 \geq \rho \geq \gamma$.

For future use, $\gamma\,(\rho)$ is the Arrow-Pratt measure of relative risk aversion for the felicity of the young (old) and σ is the curvature parameter for labor supply. Clearly, if (41) holds, i.e., the Arrow-Pratt measure of relative risk aversion for the felicity of the old is no less than that of the young, it follows from (39) that for $R > 1$, $\widehat{S}_B - \widehat{k}\widehat{L}_B < 0$

is true, i.e., (36) holds. The implication is that in a dynamically-efficient economy, there is no case for introducing a PAYG pension scheme if (41) holds.

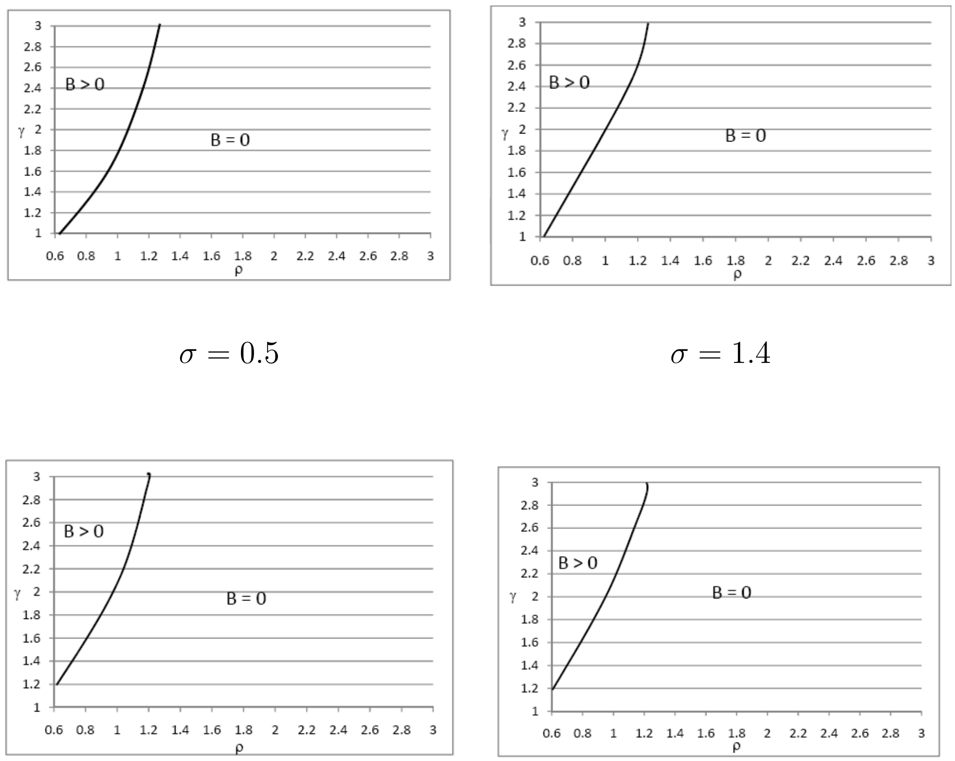
However, when $R < 1$, it follows from Lemma 6 that $\widehat{S}_B\Big|_{\mathrm{ls}} - \widehat{k}\,\widehat{L}_B\Big|_{\mathrm{ls}}$ is of indeterminate sign, and hence, it appears that satisfaction of (40) may no longer be enough to ensure (36) holds.

Proposition 3. *a) In a dynamically-efficient economy, there is no case for introducing a PAYG pension scheme if (41) holds.*
b) In a dynamically-inefficient economy, there is a case for introducing a PAYG pension scheme if (41) holds.

The upshot of the above analysis is the following. For an initially dynamically-efficient economy, there is *no* welfare rationale for introducing a PAYG pension system if the old are at least as risk averse as the young. Under that same sufficient condition, there *is* a welfare rationale for introducing a PAYG pension system if the economy is initially dynamically inefficient. In short, the Aaron-Samuelson result (both sides of it) extends to economies with elastic labor supply under a mild sufficient condition.

4.3. Numerics. We close with a numerical example, one designed to illustrate several important points. First, via this example, we wish to prove that the "intersection" of various restrictions imposed heretofore is not vacuous (i.e., a valid equilibrium does exist and exhibits the Aaron-Samuelson result). Second, we wish to underscore the point that (41) is merely a sufficient condition. For example, as the four-panel diagram below will illustrate, in a dynamically-inefficient economy, there may be a case for introducing a PAYG pension scheme even if (41) does *not* hold. Finally, we wish to leave the reader with some sense of the subsets of the parameter space for which the PAYG pension system is a good idea or not.

Example: Suppose $U(c_1, c_2, L) = \frac{1}{1-\gamma}c_1^{1-\gamma} + \beta\frac{1}{1-\rho}c_2^{1-\rho} - \frac{1}{1+\sigma}L^{1+\sigma}$, with $\beta = 1$ and $F(K, L) = 5K^{0.4}L^{0.6}$. The four-panel diagram below plots combinations of $\gamma \in [0.5, 3]$ against $\rho \in [0.5, 3]$ for which the steady-state welfare maximizing value of B is zero or positive, holding σ fixed at 0.5, 1.4, 2, and 3 respectively.



$\sigma = 0.5$ $\sigma = 1.4$

$\sigma = 2$ $\sigma = 3$

Figure 1: Combinations of ρ and γ generating $B > 0$

As Figure 1 illustrates, there is a range of values for γ and ρ – typically, $\gamma \in (1, 3)$ and $\rho \in (0.6, 1.2)$ – for which the PAYG pension system is a good idea ($B > 0$ in the pictures above). For these ranges of γ and ρ, condition (41) typically does not hold; additionally, the economy is initially dynamically inefficient, i.e., $R_{B=0} < 1$ obtains. Outside of these ranges, introducing a PAYG pension scheme is not welfare-enhancing.

5. Concluding remarks

The classic Aaron-Samuelson result argues that there is no welfare case for introducing a PAYG pension scheme if the initial steady state in a standard Diamond model (with fixed labor supply) is dynamically efficient. On the flip side, it suggests that a welfare rationale for the same exists if the initial steady state is dynamically inefficient. In this paper, we ask whether these results extend to an otherwise-identical model with endogenous labor supply. The question is of some value because the literature, employing DSGE models with numerous realistic features (such as uncertainty, altruism, incomplete markets, and so on), has delivered a "quantitative" verdict against such pensions in dynamically efficient economies; and yet, the qualitative dimensions of the issue have not been fully addressed in a simple, baseline Diamond model.

In this paper, we show that the generalization to the case with endogenous labor supply and payroll-tax financed pensions is not trivial. Asking whether there is a welfare gain from introducing a PAYG pension, we show by use of a stability condition that the answer depends on whether the initial steady state is dynamically efficient or inefficient, and on whether the capital-labor ratio increases or decreases. While savings unambiguously decrease and this goes in the direction of lowering the capital-labor ratio, labor supply, and hence employment, may also decrease, precisely because the payroll tax is distortionary. We are able to show that for an initially dynamically-efficient economy, there is no welfare rationale for introducing a PAYG pension system if the old are at least as risk averse as the young. Under that *same* condition, there is a welfare rationale for introducing a PAYG pension system if the economy is initially dynamically inefficient. In short, the Aaron-Samuelson result (both sides of it) extends to economies with elastic labor supply under a mild sufficient condition.

Appendix

A. Individual saving and labor supply

Using the first order conditions (7) and (8), it is easy to check that

$$\Gamma_{L\tau} = -w\left[u''(\cdot)(1-\tau)wL + u'(\cdot)\right] > 0 \text{ (Assumption A1)}$$

$$\Gamma_{LB} = 0$$

$$\Gamma_{Lw} = (1-\tau)\left[u''(\cdot)(1-\tau)wL + u'(\cdot)\right] > 0 \text{ (Assumption A1)}$$

$$\Gamma_{LR} = 0$$

$$\Gamma_{S\tau} = u''(\cdot)wL < 0$$

$$\Gamma_{SB} = Rv''(\cdot) < 0$$

$$\Gamma_{Sw} = -u''(\cdot)(1-\tau)L > 0$$

$$\Gamma_{SR} = RSv''(\cdot) + v'(\cdot) > 0.\text{(Assumption A2)}$$

Note the time index has been suppressed. To figure out how optimal labor supply and saving responds to the tax rate and the pension, we have from (7) and (8) that

$$\begin{bmatrix} \Gamma_{SS} & \Gamma_{SL} \\ \Gamma_{LS} & \Gamma_{LL} \end{bmatrix} \begin{bmatrix} \partial S \\ \partial L \end{bmatrix} = \begin{bmatrix} -\Gamma_{SB}\,\partial B - \Gamma_{S\tau}\partial\tau - \Gamma_{Sw}\partial w - \Gamma_{SR}\partial R \\ -\Gamma_{LB}\,\partial B - \Gamma_{L\tau}\partial\tau - \Gamma_{Lw}\partial w - \Gamma_{LR}\,\partial R \end{bmatrix}$$

from where it follows

$$L_B \equiv \frac{\partial L}{\partial B} = \frac{\begin{vmatrix} \Gamma_{SS} & -\Gamma_{SB} \\ \Gamma_{LS} & -\Gamma_{LB} \end{vmatrix}}{D} = \frac{-\Gamma_{SS}\Gamma_{LB} + \Gamma_{LS}\Gamma_{SB}}{D} = \frac{\Gamma_{LS}\Gamma_{SB}}{D} < 0$$

$$L_\tau \equiv \frac{\partial L}{\partial \tau} = \frac{\begin{vmatrix} \Gamma_{SS} & -\Gamma_{S\tau} \\ \Gamma_{LS} & -\Gamma_{L\tau} \end{vmatrix}}{D} = \frac{-\Gamma_{SS}\Gamma_{L\tau} + \Gamma_{LS}\Gamma_{S\tau}}{D}$$

$$L_w \equiv \frac{\partial L}{\partial w} = \frac{\begin{vmatrix} \Gamma_{SS} & -\Gamma_{Sw} \\ \Gamma_{LS} & -\Gamma_{Lw} \end{vmatrix}}{D} = \frac{-\Gamma_{SS}\Gamma_{Lw} + \Gamma_{LS}\Gamma_{Sw}}{D}$$

$$L_R \equiv \frac{\partial L}{\partial R} = \frac{\begin{vmatrix} \Gamma_{SS} & -\Gamma_{SR} \\ \Gamma_{LS} & -\Gamma_{LR} \end{vmatrix}}{D} = \frac{\Gamma_{LS}\Gamma_{SR}}{D}$$

where $D \equiv \Gamma_{SS}\Gamma_{LL} - \Gamma_{SL}\Gamma_{LS} > 0$ (see (9)). Also,

$$S_B \equiv \frac{\partial S}{\partial B} = \frac{\begin{vmatrix} -\Gamma_{SB} & \Gamma_{SL} \\ -\Gamma_{LB} & \Gamma_{LL} \end{vmatrix}}{D} = \frac{-\Gamma_{SB}\Gamma_{LL} + \Gamma_{SL}\Gamma_{LB}}{D} = \frac{-\Gamma_{SB}\Gamma_{LL}}{D} < 0$$

$$S_\tau \equiv \frac{\partial S}{\partial \tau} = \frac{\begin{vmatrix} -\Gamma_{S\tau} & \Gamma_{SL} \\ -\Gamma_{L\tau} & \Gamma_{LL} \end{vmatrix}}{D} = \frac{-\Gamma_{S\tau}\Gamma_{LL} + \Gamma_{SL}\Gamma_{L\tau}}{D}$$

$$S_w \equiv \frac{\partial S}{\partial w} = \frac{\begin{vmatrix} -\Gamma_{Sw} & \Gamma_{SL} \\ -\Gamma_{Lw} & \Gamma_{LL} \end{vmatrix}}{D} = \frac{-\Gamma_{Sw}\Gamma_{LL} + \Gamma_{SL}\Gamma_{Lw}}{D}$$

$$S_R \equiv \frac{\partial S}{\partial R} = \frac{\begin{vmatrix} -\Gamma_{SR} & \Gamma_{SL} \\ -\Gamma_{LR} & \Gamma_{LL} \end{vmatrix}}{D} = \frac{-\Gamma_{SR}\Gamma_{LL} + \Gamma_{SL}\Gamma_{LR}}{D} = \frac{-\Gamma_{SR}\Gamma_{LL}}{D}.$$

First note that, since $\Gamma_{LS} > 0$, and $\Gamma_{LL} < 0$,

$$\text{sign } \frac{\partial L}{\partial R} = \text{sign } \frac{\partial S}{\partial R} > 0$$

and each is positive if $\Gamma_{SR} > 0$ (Assumption A2).

1. $\frac{\partial S}{\partial \tau}$: Since $D > 0$, the sign of $\frac{\partial S}{\partial \tau}$ is the same as the sign of $(-\Gamma_{S\tau}\Gamma_{LL} + \Gamma_{SL}\Gamma_{L\tau})$ which reduces to

$$-u''(\cdot)\, wL \left[u''(\cdot)\left((1-\tau)\, w\right)^2 - d''(\cdot)\right] + \left[-u''(\cdot)\,(1-\tau)\, w\right]\left[-u''(\cdot)\,(1-\tau)\, w^2 L - u'(\cdot)\, w\right]$$

and further simplifies to

$$u''(\cdot)\, w \left[L d''(\cdot) + (1-\tau)\, w u'(\cdot)\right] < 0.$$

2. $\frac{\partial L}{\partial \tau}$: Since $D > 0$, the sign of $\frac{\partial S}{\partial \tau}$ is the same as the sign of $-\Gamma_{SS}\Gamma_{L\tau} + \Gamma_{LS}\Gamma_{S\tau}$ which reduces to

$$-\left[u''(\cdot) + R^2 v''(\cdot)\right]\left[-u''(\cdot)(1-\tau)\, w^2 L - u'(\cdot)\, w\right] + \left[-u''(\cdot)(1-\tau)\, w\right]\left[u''(\cdot)\, wL\right]$$

and further reduces to

$$= w\left[u''(\cdot)\, u'(\cdot) + R^2 v''(\cdot)\left\{u'(\cdot) + u''(\cdot)(1-\tau)\, Lw\right\}\right] < 0$$

3. $\frac{\partial L}{\partial w}$: The sign of $\frac{\partial L}{\partial w}$ is the same as the sign of $-\Gamma_{SS}\Gamma_{Lw} + \Gamma_{LS}\Gamma_{Sw}$ which reduces to

$$-\left(u''(\cdot) + R^2 v''(\cdot)\right)\left(u''(\cdot)(1-\tau)^2 wL + u'(\cdot)(1-\tau)\right) + \left(-u''(\cdot)(1-\tau)\, w\right)\left(-u''(\cdot)(1-\tau) L\right)$$

and further to

$$(1-\tau)\left[R^2 w L u''(\cdot)\, v''(\cdot)(1-\tau) - u'(\cdot)\left(R^2 v''(\cdot) + u''(\cdot)\right)\right] > 0.$$

4. $\frac{\partial S}{\partial w}$: The sign of $\frac{\partial S}{\partial w}$ is the same as the sign of $-\Gamma_{Sw}\Gamma_{LL} + \Gamma_{SL}\Gamma_{Lw}$ which is positive since $\Gamma_{Sw} > 0$, $\Gamma_{LL} < 0$, $\Gamma_{Lw} > 0$ and $\Gamma_{SL} > 0$.

B. Proof of Lemma 3

Using (27)-(28), the response of an increase in the wage rate (for fixed R and $B = 0$) on equilibrium labor supply is given by

$$\left.\widehat{L}_w\right|_{B=0} \equiv \left.\frac{\partial \widehat{L}}{\partial w}\right|_{B=0} = \frac{\begin{vmatrix} -u''(\cdot)\, w\widehat{L} - u'(\cdot) & -u''(\cdot)\, w \\ u''(\cdot)\widehat{L} & u''(\cdot) + v''(\cdot)\, R^2 \end{vmatrix}}{\begin{vmatrix} u''(\cdot)\, w^2 - d''(\cdot) & -u''(\cdot)\, w \\ -u''(\cdot)\, w & u''(\cdot) + v''(\cdot)\, R^2 \end{vmatrix}}$$
$$= \frac{-\left[u''(\cdot)\, w\widehat{L} + u'(\cdot)\right]\left[u''(\cdot) + v''(\cdot)\, R^2\right] + \left(u''(\cdot)\right)^2 w\widehat{L}}{\left[u''(\cdot)\, w^2 - d''(\cdot)\right]\left[u''(\cdot) + v''(\cdot)\, R^2\right] - \left[u''(\cdot)\, w\right]^2}$$
$$= \frac{-v''(\cdot)\, R^2\left[u''(\cdot)\, w\widehat{L} + u'(\cdot)\right] - u''(\cdot)\, u'(\cdot)}{A}.$$

where

$$A \equiv \left[u''(\cdot)\, w^2 - d''(\cdot)\right]\left[u''(\cdot) + v''(\cdot)\, R^2\right] - \left[u''(\cdot)\, w\right]^2 = R^2 v''(\cdot)\, u''(\cdot)\, w^2 - u''(\cdot)\, d''(\cdot) > 0, \tag{42}$$

Notice A is simply D evaluated at $\tau = 0$. Under Assumption A1, it follows that $\frac{\partial \widehat{L}}{\partial w} > 0$. Similarly,

$$\begin{aligned}
\left.\widehat{S}_w\right|_{B=0} &= \frac{\begin{vmatrix} u''(\cdot)\,w^2 - d''(\cdot) & -\left[u''(\cdot)\,w\widehat{L} + u'(\cdot)\right] \\ -u''(\cdot)w & u''(\cdot)\widehat{L} \end{vmatrix}}{\begin{vmatrix} u''(\cdot)\,w^2 - d''(\cdot) & -u''(\cdot)\,w \\ -u''(\cdot)\,w & u''(\cdot) + v''(\cdot)\,R^2 \end{vmatrix}} \\
&= \frac{\left[u''(\cdot)\,w^2 - d''(\cdot)\right]\left[u''(\cdot)\,\widehat{L}\right] - \left[u''(\cdot)\,w\widehat{L} + u'(\cdot)\right]u''(\cdot)\,w}{A} \\
&= \frac{u''(\cdot)\,w^2 u''(\cdot)\,\widehat{L} - d''(\cdot)\,u''(\cdot)\,\widehat{L} - u''(\cdot)\,w\widehat{L}u''(\cdot)\,w - u'(\cdot)\,u''(\cdot)\,w}{A} \\
&= \frac{-d''(\cdot)\,u''(\cdot)\,\widehat{L} - u'(\cdot)\,u''(\cdot)\,w}{A} > 0
\end{aligned}$$

The rate of return responses are as follows:

$$\begin{aligned}
\left.\frac{\partial \widehat{L}}{\partial R}\right|_{B=0} &= \frac{u''(\cdot)\,w\left[-v'(\cdot) - Rv''(\cdot)\,\widehat{S}\right]}{A} \\
\left.\frac{\partial \widehat{S}}{\partial R}\right|_{B=0} &= \frac{\left[-v'(\cdot) - Rv''(\cdot)\,\widehat{S}\right]\left[u''(\cdot)\,w^2 - d''(\cdot)\right]}{A}
\end{aligned}$$

from where it is clear that

$$\text{sign}\ \frac{\partial \widehat{L}}{\partial R} = \text{sign}\ \frac{\partial \widehat{S}}{\partial R} = \text{sign}\ \left[v'(\cdot) + R\widehat{S}v''(\cdot)\right]_{B=0} = \text{sign}\ \left[1 - \Phi_v^0\right]$$

Hence,

$$\Phi_v^0 \leq 1 \Longleftrightarrow \text{sign}\ \frac{\partial \widehat{L}}{\partial R} = \text{sign}\ \frac{\partial \widehat{S}}{\partial R} \geq 0.$$

Finally, the benefit responses are as follows:

$$\left.\widehat{L}_B\right|_{B=0} = \frac{\begin{vmatrix} u''(\cdot)\, w + u'(\cdot)\, \frac{1}{\widehat{L}} & -u''(\cdot)\, w \\ -\left[u''(\cdot) + Rv''(\cdot)\right] & \left[u''(\cdot) + v''(\cdot)\, R^2\right] \end{vmatrix}}{A}$$

$$= \frac{\left[u''(\cdot)\, w + u'(\cdot)\, \frac{1}{\widehat{L}}\right]\left[u''(\cdot) + v''(\cdot)\, R^2\right] - \left[u''(\cdot)\, w\right]\left[u''(\cdot) + Rv''(\cdot)\right]}{A}$$

$$\left.\widehat{S}_B\right|_{B=0} = \frac{\begin{vmatrix} u''(\cdot)\, w^2 - d''(\cdot) & u''(\cdot)\, w + u'(\cdot)\, \frac{1}{\widehat{L}} \\ -u''(\cdot)\, w & -\left[u''(\cdot) + Rv''(\cdot)\right] \end{vmatrix}}{A}$$

$$= \frac{-\left[u''(\cdot)\, w^2 - d''(\cdot)\right]\left[u''(\cdot) + Rv''(\cdot)\right] + \left[u''(\cdot)\, w + u'(\cdot)\, \frac{1}{\widehat{L}}\right]\left[u''(\cdot)\, w\right]}{A}$$

The latter expressions are easily reduced to

$$\left.\widehat{L}_B\right|_{B=0} = \frac{u'(\cdot)\, \frac{1}{\widehat{L}}\left[u''(\cdot) + v''(\cdot)\, R^2\right] + v''(\cdot)\, u''(\cdot)\, wR\,(R-1)}{A} < 0 \text{ if } R < 1. \quad (43)$$

and

$$\left.\widehat{S}_B\right|_{B=0} = \frac{d''(\cdot)\left[u''(\cdot)+Rv''(\cdot)\right]-Rv''(\cdot)\left[u''(\cdot)w^2+wRu'(\cdot)\frac{1}{L}\right]+wu'(\cdot)\frac{1}{L}\left[u''(\cdot)+v''(\cdot)R^2\right]}{A} \tag{44}$$
$$= \frac{d''(\cdot)\left[u''(\cdot)+Rv''(\cdot)\right]-Rv''(\cdot)u''(\cdot)w^2+wu'(\cdot)\frac{1}{\widehat{L}}u''(\cdot)}{A} < 0.$$

C. Proof of Lemma 4

The characteristic polynomial is given by $p(\lambda) \equiv \lambda^2 + \lambda A_1 + A_0$. Stability (saddle-point) requires one characteristic root to satisfy $|\lambda_1| < 1$ and the other, $|\lambda_2| > 1$. Evaluated for $\lambda = 1$, we get

$$p(1) = 1 + A_1 + A_0 = 1 + \frac{\left[\left[\widehat{L} - \widehat{k}\ \widehat{L}_w k R_k\right] - \widehat{S}_R R_k\right]}{\widehat{k}\ \widehat{L}_R R_k} + \frac{\widehat{S}_w}{\widehat{L}_R}$$

which, after routine simplification yields,

$$p(1) = \frac{1}{\widehat{k}\ \widehat{L}_R R_k}\left[\widehat{L} + \left[\ \widehat{L}_R - \widehat{k}\ \widehat{L}_w\right] k R_k - \widehat{S}_R R_k + \widehat{S}_w k R_k\right].$$

Notice $p(-1) = 1 - A_1 + A_0 > 0$. Then, saddle path stability requires $p(1) < 0$ or that

$$1 + A_1 + A_0 = \frac{1}{\widehat{k}\ \widehat{L}_R R_k}\left[\widehat{L} + \left[\widehat{L}_R - \widehat{k}\ \widehat{L}_w\right]\widehat{k} R_k - \widehat{S}_R R_k + \widehat{S}_w \widehat{k} R_k\right] < 0.$$

Since $\widehat{L}_R > 0$ holds (see (31)), saddle path stability requires

$$\left[\widehat{L} + \left[\widehat{L}_R - \widehat{k}\ \widehat{L}_w\right]\widehat{k} R_k - \widehat{S}_R R_k + \widehat{S}_w \widehat{k} R_k\right] > 0$$

hold. For future use, note that using $w_k = -\widehat{k} R_k$, this last condition may be written as

$$\widehat{L} - \widehat{L}_R w_k + \widehat{k} w_k \, \widehat{L}_w - \widehat{S}_R R_k - \widehat{S}_w w_k > 0. \tag{45}$$

It follows from the definition of $\phi(k)$ that

$$\phi'(k) = 1 - \frac{\left(\widehat{S}_w w_k + \widehat{S}_R R_k\right) \widehat{L} - \widehat{S}\left(\widehat{L}_w w_k + \widehat{L}_R R_k\right)}{\left(\widehat{L}\right)^2} > 0.$$

Using $\widehat{S} \equiv \widehat{K}$ and $\widehat{k} \equiv \widehat{K}/\widehat{L}$, we can rewrite this as

$$\widehat{L} - \left(\widehat{S}_w w_k + \widehat{S}_R R_k\right) + \widehat{L}_w \widehat{k} w_k - \widehat{L}_R w_k > 0$$

the same as (45).

D. Proof of Lemma 5

Noting that $\widehat{k} \equiv \widehat{K}/\widehat{L}$, it follows that

$$\frac{\partial \widehat{k}}{\partial B} = \frac{\widehat{k}}{\widehat{K}} \frac{\partial \widehat{K}}{\partial B} - \frac{\widehat{k}}{\widehat{L}} \frac{\partial \widehat{L}}{\partial B}$$

and

$$\widehat{L} \frac{\partial \widehat{k}}{\partial B} = \left[\frac{\partial \widehat{K}}{\partial B} - \widehat{k} \frac{\partial \widehat{L}}{\partial B}\right].$$

Since $\widehat{K} \equiv \widehat{S}(B, w\left(\widehat{k}\right), R\left(\widehat{k}\right))$, we have

$$\frac{\partial \widehat{K}}{\partial B} = \widehat{S}_B + \left[\widehat{S}_w w_k + \widehat{S}_R R_k\right] \frac{\partial \widehat{k}}{\partial B}$$

and since $\widehat{L} \equiv \widehat{L}\left(B, w\left(\widehat{k}\right), R\left(\widehat{k}\right)\right)$, we have

$$\frac{\partial \widehat{L}}{\partial B} = \widehat{L}_B + \left[\ \widehat{L}_w w_k + \ \widehat{L}_R R_k\right] \frac{\partial \widehat{k}}{\partial B}.$$

Then,

$$\widehat{L}\frac{\partial k}{\partial B} = \frac{\partial \widehat{K}}{\partial B} - \widehat{k}\frac{\partial \widehat{L}}{\partial B} = \widehat{S}_B + \left[\widehat{S}_w w_k + \widehat{S}_R R_k\right] \frac{\partial \widehat{k}}{\partial B} - \widehat{k}\left(\widehat{L}_B + \left[\ \widehat{L}_w w_k + \ \widehat{L}_R R_k\right]\right) \frac{\partial \widehat{k}}{\partial B}$$
$$\Leftrightarrow \frac{\partial \widehat{k}}{\partial B} = \frac{\widehat{S}_B - \widehat{k}\ \widehat{L}_B}{\widehat{L} - \left[\widehat{S}_w w_k + S_R R_k\right] + \widehat{k}\left[\widehat{L}_w w_k + \ \widehat{L}_R R_k\right]}.$$

From (32), we have

$$\widehat{L} + \left[\widehat{L}_R - \widehat{k}\ \widehat{L}_w\right] \widehat{k} R_k - \widehat{S}_R R_k + \widehat{S}_w \widehat{k} R_k > 0.$$

Using $w_k = -\widehat{k} R_k$, we can rewrite the previous condition as

$$\widehat{L} - \widehat{S}_R R_k - \widehat{S}_w w_k - \ \widehat{L}_R w_k + \widehat{k} L_w w_k > 0 \Rightarrow \widehat{L} - \left(\widehat{S}_R R_k + \widehat{S}_w w_k\right) + \widehat{k}\left[\widehat{L}_R R_k + \widehat{L}_w w_k\right] > 0$$

the same as the denominator of $\frac{\partial \widehat{k}}{\partial B}$. It follows that at a saddle-point stable $\widehat{k}$,

$$\text{sign}\left[\frac{\partial \widehat{k}}{\partial B}\right] = \text{sign}\left[\widehat{S}_B - \widehat{k}\widehat{L}_B\right]$$

E. Proof of Proposition 2

If the pension is financed by a lump-sum tax levied on the young, the first order condition to the individual optimization problem can be written as

$$u'\left(w_t\,\widehat{L}_t - B - \widehat{S}_t\right)[w_t] - d'\left(\widehat{L}_t\right) = 0$$
$$-u'\left(w_t\,\widehat{L}_t - B - \widehat{S}_t\right) + R_{t+1}v'\left(R_{t+1}\widehat{S}_t + B\right) = 0.$$

Note that the first order conditions are evaluated in equilibrium making use of $T = B$. Hence, totally differentiating yields

$$u''(\cdot)\left[\,\widehat{L}\partial w + w\partial\widehat{L} - \partial B - dS\right]w_t + u'(\cdot)\partial w - d''(\cdot)\partial\widehat{L} = 0$$
$$-u''(\cdot)\left[\,\widehat{L}\partial w + wd\widehat{L} - \partial B - \partial\widehat{S}\right] + \partial R v'(\cdot) + Rv''(\cdot)\left[\widehat{S}\partial R + R\partial\widehat{S} + \partial B\right] = 0$$

Evaluating these expressions for $B = 0$ yields

$$\left[u''(\cdot)w^2 - d''(\cdot)\right]\partial\widehat{L} + \left[-u''(\cdot)w\right]\partial\widehat{S} = \left[-u''(\cdot)w\widehat{L}\right]\partial w + \left[u''(\cdot)w\right]\partial B$$
$$\left[-u''(\cdot)w\right]\partial\widehat{L} + \left[u''(\cdot) + R^2v''(\cdot)\right]\partial\widehat{S} = u''(\cdot)\widehat{L}\partial w - \left[u''(\cdot) + Rv''(\cdot)\right]\partial B + \left[-v'(\cdot) - Rv''(\cdot)\widehat{S}\right]\partial R.$$

Then,

$$\left.\frac{\partial \widehat{L}}{\partial B}\right|_{\text{ls}} = \frac{\begin{vmatrix} u''(\cdot)w & -u''(\cdot)w \\ -\left[u''(\cdot)+Rv''(\cdot)\right] & \left[u''(\cdot)+v''(\cdot)R^2\right] \end{vmatrix}}{\begin{vmatrix} u''(\cdot)w^2 - d''(\cdot) & -u''(\cdot)w \\ -u''(\cdot)w & u''(\cdot)+v''(\cdot)R^2 \end{vmatrix}}$$

which may be simplified to yield

$$\left.\frac{\partial \widehat{L}}{\partial B}\right|_{\text{ls}} = \frac{u''(\cdot)wv''(\cdot)R\left(R-1\right)}{A} \lesseqgtr 0 \text{ if } R \lesseqgtr 1$$

since $A > 0$ (see (42)). Similarly,

$$\left.\frac{\partial \widehat{S}}{\partial B}\right|_{\text{ls}} = \frac{\begin{vmatrix} u''(\cdot)w^2 - d''(\cdot) & u''(\cdot)w \\ -u''(\cdot)w & -\left[u''(\cdot)+Rv''(\cdot)\right] \end{vmatrix}}{\begin{vmatrix} u''(\cdot)w^2 - d''(\cdot) & -u''\left(\cdot\right)w \\ -u''(\cdot)w & u''\left(\cdot\right)+v''\left(\cdot\right)R^2 \end{vmatrix}},$$

which may be simplified to yield

$$\left.\frac{\partial \widehat{S}}{\partial B}\right|_{\text{ls}} = \frac{d''(\cdot)\left[u''(\cdot)+Rv''(\cdot)\right] - u''(\cdot)w^2Rv''(\cdot)}{D} < 0.$$

Hence, under lump-sum taxation we have

$$\left.\frac{\partial \widehat{L}}{\partial B}\right|_{\text{ls}} \lesseqgtr 0 \text{ if } R \lesseqgtr 1, \quad \left.\frac{\partial \widehat{S}}{\partial B}\right|_{\text{ls}} < 0.$$

F. Proof of Lemma 6

Using (44)-(43), and the expressions derived in the proof of Proposition 2, we have

$$\begin{aligned}
\frac{\partial \widehat{L}}{\partial B} &= \left.\frac{\partial \widehat{L}}{\partial B}\right|_{\mathrm{ls}} + \frac{u'(\cdot)\frac{1}{\widehat{L}}\left[u''(\cdot)+v''(\cdot)R^2\right]}{A} < \left.\frac{\partial \widehat{L}}{\partial B}\right|_{\mathrm{ls}}, \\
\frac{\partial \widehat{S}}{\partial B} &= \left.\frac{\partial \widehat{S}}{\partial B}\right|_{\mathrm{ls}} + \frac{u'(\cdot)\frac{1}{\widehat{L}}\left[u''(\cdot)w\right]}{A} < \left.\frac{\partial \widehat{S}}{\partial B}\right|_{\mathrm{ls}},
\end{aligned}$$

where $A > 0$.

G. Proof of Lemma 7

It is easy to rewrite (40) as

$$u'(\cdot)\frac{1}{\widehat{L}}\left\{u''(\cdot)\left(w-\widehat{k}\right) - \widehat{k}v''(\cdot)R^2\right\}$$

noting $c_1 = (w-\widehat{k})\widehat{L}$ and $c_2 = \widehat{k}R\widehat{L}$, this is rewritten as

$$u'(\cdot)\frac{1}{\left(\widehat{L}\right)^2}\left\{u''(\cdot)c_1 - \widehat{k}\widehat{L}v''(\cdot)R^2\right\}$$

and further as

$$u'(\cdot)\frac{1}{\left(\widehat{L}\right)^2}u'(\cdot)\left\{\frac{u''(\cdot)c_1}{u'(\cdot)} - \frac{c_2 v''(\cdot)}{u'(\cdot)}R\right\}.$$

The rest is immediate once (8) is invoked.

H. Proof of Proposition 3

b) Using (44)-(43), it is possible to re-write $\widehat{S}_B - \widehat{k}\widehat{L}_B < 0$ as

$$\frac{1}{\widehat{L}}\left\{u'(\cdot)\,\widehat{k}\left[u''(\cdot)+v''(\cdot)\,R^2\right]-u'(\cdot)\,wu''(\cdot)\right\}+Rv''(\cdot)\,u''(\cdot)\,w^2+\widehat{k}v''(\cdot)\,u''(\cdot)\,wR\,(R-1)>d''(\cdot)\left[u''(\cdot)\right. \tag{46}$$

Under the condition, $\Phi_v^0 \leq \Phi_u^0$, the first term (in parenthesis) on the l.h.s of (46) is positive. The second term on the l.h.s of (46) is positive. When $R < 1$, the third term on the l.h.s of (46) is negative. Also, the r.h.s of (46) is negative. Thus, under (41), for $\widehat{S}_B - \widehat{k}\widehat{L}_B < 0$, it is sufficient that

$$Rv''(\cdot)\,u''(\cdot)\,w^2+\widehat{k}v''(\cdot)\,u''(\cdot)\,wR\,(R-1)>0.$$

It is easy to show that

$$Rv''(\cdot)\,u''(\cdot)\,w^2+\widehat{k}v''(\cdot)\,u''(\cdot)\,wR\,(R-1)=Rwv''(\cdot)\,u''(\cdot)\left[\left(w-\widehat{k}\right)+R\widehat{k}\right]>0$$

since $c_1 = (w - \widehat{k})\widehat{L} > 0$.

Notes

[1]*The authors thank an associate editor, Brent Kreider, Xiying Liu and Donggyu Yi for helpful comments, and Marcelo Oviedo, Rajesh Singh and Min Wang for help with some of the computations. We also acknowledge financial support from the Danish Council for Independent Research (Social Sciences) under the Danish Ministry of Science, Technology and Innovation.

[1]Indeed, among academic economists, a verdict against unfunded pension systems has come to be generally accepted. Fuster, Imrohoroglu, and Imrohoroglu (2007) describe the overall debate, and the dénoument as follows: "The unfunded public pension system provides insurance against mortality and individual income risks for which insurance through private markets is either unavailable or difficult due to moral hazard and other reasons. At the same time, the unfunded system distorts the saving and labour supply decisions and imposes a deadweight cost on the society. When these two sets of effects of social security are evaluated in economic models, it is almost always the case that the unfunded system has an overall welfare cost on the households."

[2]It is well-known that introduction of a PAYG pension scheme bestows the initial old generation with a "gift". These transitional gains have to be weighed against the more permanent consequences of such a scheme. In DSGE, quantitatively-oriented models, such as Auerbach and Kotlikoff (1987), Conesa and Krueger (1999), Conesa and Garriga (2003), and Fuster, Imrohoroglu, and Imrohoroglu (2007), these transi-

tional gains are given substantial prominence. In the following, we follow the "qualitative" literature derived from Breyer and Straub (1993) and confine our analysis to steady-state effects.

[3]For example, the role played by unfunded pension systems in ameliorating idiosyncratic risks (such as, those involving mortality or labor income) in worlds with incomplete financial markets has been highlighted and surveyed in Krueger (2006). Andersen and Bhattacharya (2011) and Caliendo and Gahramanov (2010) study the importance of agent myopia in providing a justification for such pension systems. Fuster, Imrohoroglu, and Imrohoroglu (2007) introduce bi-directional altruism along with mortality and earnings risks in a framework similar to one adopted by Conesa and Krueger (1999). Cooley and Soares (1999) explore a political-economy justification for PAYG pensions – see Galasso and Profeta (2004) for a survey.

[4]Life-cycle models with endogenous labor supply have been used extensively in the social security literature (see, for example, Auerbach and Kotlikoff (1987), Homburg (1990), Breyer and Straub (1993), and Heijdra and Mierau (2010).

[5]As noted earlier, the literature, employing DSGE models with simultaneous, multiple realistic features (such as uncertainty, altruism, incomplete markets, and so on), has delivered a "quantitative" verdict against PAYG pensions in dynamically-efficient economies. However, the qualitative dimensions of the issue (especially the analytics of the extension to elastic labor supply) have not been fully addressed in a simple, baseline Diamond model.

[6]A version of this result appears in Breyer and Straub (1993).

[7]In places below, we consider as a benchmark, a lump-sum tax on wage income. In that case, if T is the lump-sum tax, the analog of (4) is $T = B$.

[8]Breyer and Straub (1993) consider three different formulations of the PAYG scheme: a) fixed contribution rate τ but pensions are tied to past labor supply, b) time-varying contribution rate, and c) lump-sum contributions and benefits.

[9]There is little consensus about whether the coefficient of relative risk aversion is less than or greater than one. The usual values for these utility parameters that are popular in the literature – see Meyer and Meyer (2005) for instance – are estimated, either from static, micro studies or for time horizons of a quarter or less. Otrok (2001) in a study of the welfare cost of business cycles estimates it to be 0.72 with a 90% confidence interval of [0.54, 0.95]. Five out of six of Hansen and Singleton's (1983) estimates, ranging from 0.17 to 1.36, are also smaller than one.

[10]In a small open economy, $R = f'(k)$ is fixed, implying k is fixed; hence, if $R < 1$, $U'(0) > 0$ clearly holds.

[11]See footnote 7.

[12]Notice that (30) had already imposed $1 \geq \Phi_v^0$, and $1 \geq \Phi_u^0$.

[13]Halek and Eisenhauer (2001) use survey data from the University of Michigan Health and Retirement Study. Their estimation procedure resulted in 2,376 household-level observations of the relative risk aversion parameter with a median

value of 0.88. Notably, they find that "being age 65 or older significantly increases one's risk aversion by 36.22 percent to 38.26 percent." (pg. 14). Salm (2010) studies 583 elderly (over-65) respondents combining data from waves five and six of the Health and Retirement Study (HRS) and finds a coefficient of relative risk aversion for the old of about 0.55.

References

[1] Aaron, Henry. (1966). The Social Insurance Paradox. *Canadian Journal of Economics*, 32, 371-4

[2] Abel, Andrew B., N. Gregory Mankiw, Lawrence H. Summers and Richard J. Zeckhauser (1989) Assessing dynamic efficiency: theory and evidence. *Review of Economic Studies* 56, 1–19

[3] Andersen, Torben M. and Joydeep Bhattacharya (2011) On myopia as rationale for social security. *Economic Theory* 47, 135–158

[4] Auerbach, Alan J. and Lawrence. J. Kotlikoff. (1987) *Dynamic Fiscal Policy.* New York, NY, USA: Cambridge University Press.

[5] Barbie, Martin, Marcus Hagedorn and Ashok Kaul (2004) Assessing aggregate tests of efficiency for dynamic economies. *Topics in Macroeconomics*: 4(1), Article 16.

[6] Blake, David (2006) *Pension Economics*, Wiley, Hoboken, NJ

[7] Blanchard, O., and S. Fischer (1989) *Lectures on Macroeconomics*, Cambridge, MA: MIT Press.

[8] Breyer, Friedrich and Martin Straub. (1993). Welfare effects of unfunded pension systems when labor supply is endogenous," *Journal of Public Economics*, 50(1), 77-91.

[9] Caliendo, Frank N. and Emin Gahramanov (2011) Myopia and pensions in general equilibrium. *Journal of Economics and Finance* DOI: 10.1007/s12197-011-9187-6

[10] Cazzavillan, Guido. and Patrick A. Pintus (2004) Robustness of multiple equilibria in OLG economies. *Review of Economic Dynamics* 7, 456–475.

[11] Conesa, Juan C. and D. Krueger (1999) Social Security Reform with heterogeneous Agents. *Review of Economics Dynamics* 2, 757-795.

[12] Conesa, Juan C. and Carlos Garriga (2003) Status quo problem in social security reforms. *Macroeconomic Dynamics* 7, 691-710 doi:10.1017/S1365100502020217

[13] Cooley, Thomas F. and Jorge Soares. (1999) A positive theory of social security based on reputation. *Journal of Political Economy*, 107(1), 135-160.

[14] Diamond, Peter A. (1965) National debt in a neoclassical growth model. *American Economic Review*, 55(5), 1126-1150.

[15] Fuster, Luisa., Ayse Imrohoroglu and Selahattin Imrohoroglu. (2007) Elimination of social security in a dynastic framework. *Review of Economic Studies* 74 (1), 113-145.

[16] Galasso, Vincenzo and Paola Profeta (2004) Lessons for an aging aociety: the political sustainability of social security systems. *Economic Policy*, 63-115.

[17] Halek, Martin and Joseph G. Eisenhauer (2001) Demography of risk aversion. *Journal of Risk and Insurance*, 68(1), 1-24.

[18] Hansen, Lars Peter and Kenneth J. Singleton (1983) Stochastic consumption, risk aversion, and the temporal return of asset returns. *Journal of Political Economy* 91, 249-65.

[19] Heijdra, Ben J. and Jochen O. Mierau. (2010) Growth effects of consumption and labor-income taxation in an overlapping-generations life-cycle model 14, 151-175

[20] Homburg, Stefan (1990). The efficiency of unfunded pension schemes. *Journal of Institutional and Theoretical Economics* 146, 640-647.

[21] Krueger, Dirk (2006). Public insurance against idiosyncratic and aggregate risk: the case of social security and progressive taxation. *CESifo Economic Studies* 52, 587-620.

[22] Lopez-Garcia, Miguel-Angel (2008) On the role of public debt in an OLG model with endogenous labor supply. *Journal of Macroeconomics* 30, 1323–1328.

[23] Meyer, Donald J. and Jack Meyer (2005) Relative risk aversion: what do we know? *Journal of Risk and Uncertainty* 31, 243–262.

[24] (2001) Stability of equilibria in the overlapping generations model with endogenous labor supply. *Journal of Economic Dynamics and Control* 25, 1647-1663.

[25] Nourry, Carine and Venditti, Alain. (2006) Overlapping Generations Model with Endogenous Labor Supply: General Formulation. *Journal of Optimization Theory and Applications*, 128 (2), 355–377.

[26] Otrok, Chris. (2001) On measuring the welfare cost of business cycles. *Journal of Monetary Economics* 47, 61-92.

[27] Salm, Martin. (2010) Subjective mortality expectations and consumption and saving behaviours among the elderly. *Canadian Journal of Economics*, 43, 1040–1057.

[28] Samuelson, Paul A. (1958) An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy* 66 (6), 467-482.

[29] Samuelson, Paul A. (1975) Optimum social security in a life-cycle growth model, *International Economic Review* 16(3), 539-544.